THE RING OF ENDOMORPHISMS OF A FINITE DIMENSIONAL MODULE

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ABSTRACT

Let S denote the ring of endomorphisms of a finite dimensional module M_R. Necessary and sufficient conditions for a nil subring of S to be nilpotent are given. We place conditions on M_R so that every nil subring of S will be nilpotent.

1. Introduction.

Assume that a module M contains a direct sum of n-nonzero summands and the number of nonzero summands of any other direct sum of M is at most n. We call M a *finite dimensional* module [3] and say that M has *dimension* n, written $\dim M = n$. Throughout this note R always denotes a ring, M a right R-module and S the ring of endomorphisms of M_R. We define $N(S) = \{x \in S\colon \ker x$ is an essential submodule of $M\}$. If x and y are in S then $(xy)(m) = x(y(m))$ where m is in M.

Feller [2] has studied the relationship between M and S when $\dim M = 1$. If M is a uniform module (that is, $\dim M = 1$) then $S/N(S)$ is an integral domain. If in addition M is Noetherian then $N(S)$ is a nil ideal of S (th. 3.1 [2]). This paper extends these two results. If M is a finite dimensional module then $S/N(S)$ is embeddable in a completely reducible ring. A nil subring of $S/N(S)$ is nilpotent. Also, a nil subring K of S is nilpotent if and only if $K \cap N(S)$ is nilpotent (Theorem 3). In Section 3 we place a condition on M so that $N(S)$ will be nilpotent (Theorem 10). The Noetherian condition implies our condition but not conversely.

2. The nil structure of S

We denote the injective hull of M_R by $I(M)$. Recall, $\dim I(M) = \dim M$ whenever M is finite dimensional. For a general reference consult [7].

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NOTATION. Let H denote the ring of endormorphions of $I(M)$. Equate $N(H) = \{h \in H\colon \ker h$ is an essential submoduse of $M\}$.

The first part of the proposition below is well-known.

PROPOSITION 1. *Let M be a finite dimensional module. Then the ring H of endomorphisms of $I(M)$ is semiperfect. Furthermore, $\dim M = H/N(H)$.*

PROOF. We refer to the proof of prop. 2 of [7, p. 103]. If the dimension of H/N is k then there will be k primitive idempotents $e_1, e_2, \cdots e_k$ in H/N. These idempotents remain primitive in H. Hence, $I(M) = e_1 I(M) + \cdots + e_k I(M)$ where each summand is injective and is indecomposable and thus has dimension 1. Therefore, $k = \dim M$.

Recall, S is the ring of endomorphisms of M_R and H the ring of endomorphisms of $I(M)$. Let S' denote the set of elements h of H such that the restriction of h to M is in S. Then $S/N(S)$ is ring isomorphic to $S'/N(S')$; $x + N(S)$ is mapped onto $x' + N(S')$ where the restriction of x' to M is equal to x. Also the map $g(y + N(S')) = y + N(N)$ where $y \in S'$, is an embedding of $S'/N(S')$ into $N/H(H)$, since $N(S) = N(H) \cap S'$. Summing up, $S/N(S)$ is ring isomorphic to a subring of $H/N(H)$.

LEMMA 2. *If S is a ring of endomorphisms of a module M, then $S/N(S)$ is embeddable in the regular ring $H/N(H)$ where H is the ring of endomorphisms of $I(M)$.*

PROOF. By the above paragraph, $S/N(S)$ is embedded in $H/N(H)$. The ring $H/N(H)$ is regular in the sense of von Neumann [7, p. 102].

If A is nonempty subset of a ring R we define $r(A) = \{x \in R\colon ax = 0$ for all $a \in A\}$. This right ideal is called the *right annihilator* of A in R. The left annihilator of A is similarly defined.

THEOREM 3. *Let S be the ring of endomorphisms of a finite dimensional module with dimension n. Then $S/N(S)$ is embeddable in a completely reducible ring of dimension n. A chain of right (or left) annihilators in $S/N(S)$ has at most n nonzero terms. Furthermore, a nil subring of $S/N(S)$ is nilpotent and has an index of nilpotentcy of at most $n + 1$. If $N(S)$ is nilpotent with index k, than a nil subring of S is nilpotent with index at most $k(n + 1)$.*

PROOF. Let S denote the ring of endomorphisms of M, H the ring of endomorphisms of $I(M)$. Let $\dim M = n$. By Lemma 2 the quotient ring $S/N(S)$ is embedded in $H/N(H)$. If A and B are nonempty subsets of $S/N(S)$ and $r(A) \subset r(B)$ in $S/N(S)$ then $r(A) \subset r(B)$ in $H/N(H)$. Thus, a chain of right annihilators in $S/N(S)$ of length k forces a chain of right ideals of length k in $H/N(H)$. A chain of right ideals of $H/N(H)$ has at most n nonzero terms because $H/N(H)$ is a completely reducible ring and has dimension n by Proposition 1. Therefore, a chain of right (or left) annihilators in $S/N(S)$ has at most n nonzero terms. A nil subring of $H/N(H)$ is nilpotent and has an index of nilpotency of at most $n+1$. The remainig part is clear.

COROLLARY 4. (*Feller* [2]). *If M_R has dimension* 1 *then $S/N(S)$ is an integral domain.*

PROOF. A completely reducible ring with dim 1 is a division ring. The result is clear.

If $M = R$ then $S = R$ and $N(S) = N(R)$. Clearly, $N(R)$ is the right singular ideal of R [6].

COROLLARY 5. (*Shock* [10]). *Let R be a right finite dimensional ring with dim n. Then a nil subring K of R is nilpotent if and only if $K \cap N(R)$ is nilpotent. If $N(R)$ is nilpotent with index k then every nil subring of R is nilpotent and has index of nilpotency of at most $k(n+1)$.*

PROOF. The proof follows directly from Theorem 3.

If a ring R does not have an identity 1, embed it into a ring with 1, written R_1[1, p. 10]. Thus, $\mathrm{Hom}(M_R, M_R) = \mathrm{Hom}(M_{R_1}, M_{R_1})$, M_R is injective if and only if $M^*_{R_1}$ is injective, and $\dim M_R = \dim M^*_{R_1}$ [1, p. 10–11]. Also $R/N(R)$ is embedded in $R_1/N(R_1)$. Theorem 3 is valid for rings without 1.

3. Nil subrings of S are nilpotent.

In this section we place conditions on M so that every nil subring of S will be nilpotent. Our approach is via Theorem 3. If M is finite dimensional and $N(S)$ is nilpotent then every nil subring of S is nilpotent.

Let S be the ring of endomorphisms of the module M_R. Then ${}_SM_R$ is a bimodule. For nonempty subset A of M we define $r(A) = \{x \in R\colon ax = 0 \text{ for all } a \in A\}$. The right ideal $r(A)$ is called an *annihilator of M*. We say that the module M_R satisfies the *maximum condition on annhihlators* provided that every nonempty subcollection of $\{r(A)\colon A \text{ is nonempty subset of } M\}$ has a maximal element. We define $l(A) = \{p \in S\colon pa = 0 \text{ for all } a \in A\}$. In like manner we make similar definitions for the module ${}_SM$ and ${}_SS$ and speak of the *minimum condition on annihilators* of ${}_SM$ and ${}_SS$. Also if K is a nonempty subset of S we equate KM with $\{km\colon$ for all $m \in M$ and for all $k \in K\}$.

LEMMA 6. *If sM satisfies the minimum (maximum) condition on annihilators, then so does $_SS$.*

PROOF. Let A and B be nonempty subsets of S with $A \subset B$. Then $l(A) \supset l(B)$ in $_SS$ implies $AM \subset BM$ in M and $l(AM) \supset l(BM)$ in S. The proof is clear.

PROPOSITION 7. *(Mewborn and Winton). Let S be the ring of endomorphisms of a module M. If sM satisfies the minimum condition on annihilators then N(S) is nilpotent.*

PROOF. By the lemma $_SS$ satisfies the minimum condition on annihilators. This implies that $_SS$ satisfies the maximum condition on annihilators. Let $N = N(S)$. We have $r(N^j) = r(N^{j+1})$ in S for positive integer j. Let $x \in N - r(N^j)$. If $N^j yx = 0$ for all $y \in N$ then $x \in r(N^{j+1})$, a contradiction. Thus, $x_1 x \in N - r(N^j)$ for some x_1 in N. We continue in this manner to construct an infinite sequence $x, x_1, x_2, \cdots$ in N with $x_k \cdots x_1 x \neq 0$ for all positive integers k. Therefore, ker $x \subset$ ker $x_1 x \subset$ ker $x_2 x_1 x \subset \cdots$ because ker x_{k+1} is essential in M and meets $x_k \cdots x_1 xM$ for all k. This implies $l(\ker x) \supset l(\ker x_1 x) \supset \cdots$ in S, a contradiction. We conclude that $N \subset r(N^j)$ and N is nilpotent.

COROLLARY 8. *(Mewborn and Winton [9]). If a ring R satisfies the maximum condition on annihilators of R_R then the right singular ideal is nilpotent.*

PROOF. The proof is clear.

Let A and B be submodules of M_R. Following [8] we say that B is an *M-rational extension* of A provided that $A \subset B$ and if f is any R-homomorphism from a submodule of B into M and the kernel of $f \supseteq A$ then f must be the zero map. Also, a submodule A of B is said to be *M-rationally closed* if A has no proper M-rational extensions in B. A right ideal T of R is said to be *M-dense* provided that $mT = 0$ with m in $I(M)$ implies $m = 0$. Equivalently, for each $0 \neq m$ in M and x in R there exists y in R such that $my \neq 0$ and xy in T. If $b \in M$ and P is a nonempty subset of M we define $b^{-1}P = \{r \in R: br \in P\}$. For a submodule K of M let $K' = \{m \in M: m^{-1}K$ is M-dense$\}$ and K' is a submodule and is called the *rational closure* of K in M [11]. Furthermore, K is M-rationally closed if and only if $K = K'$ [11]. We say that M_R satisfies the *maximum condition on rationally closed submodule* if every nonempty subcollection of the collection of M-rationally closed submodules of M has a maximal element. If $M = R$, then a R-rationally closed right ideal of R is an annihilator of $I(R)$ (and conversely) [11]. Rings with the maximum condition on rationally closed right ideals are called

Solid Goldie rings [5]. A Noetherian ring is a Solid Goldie ring but not conversely In a Solid Goldie ring a nil subring is nilpotent; our Theorem 10 below generalizes this fact.

LEMMA 9. *Let A and B be submodules of M_R with $A \subset B$. Then $A' \subset B'$ if and only if $b^{-1}A$ is not M-dense for some $b \in B - A$.*

PROOF. Clearly, $A' = B'$ if and only if $B \subset A'$ if and only if $b^{-1}A$ is M-dense for all $b \in B - A$.

THEOREM 10. *Assume that M_R satisfies the maximum condition on rationally closed submodules of M. If S is the ring of endomorphisms of M_R then ${}_SM$ satisfies the minimum condition on anihilators. Furthermore, every nil subring of S is nilpotent.*

PROOF. We claim that M_R is finite dimensional. Suppose that $A + B$ is a direct sum of nonzero submodules of M. Then for any $b \in B - A$ we have $b(b^{-1}A) = (0)$ and $b^{-1}A$ is not *M-dense* and thus $A' \subset (A + B)'$ by Lemma 9. So an infinite direct sum of nonzero submodules of M would force an increasing sequence of rationally closed submodules of M, a contradiction. Therefore, M is finite dimensional. Let E and F be nonempty subsets of M with $E \subset F$ and suppose $l(E) \supset l(F)$. Since $l(E)$ is equal to the left annihilator of the submodule of M_R generated by E we can regard E (and F) as submodules. There is $x \in S$ such that $xE = 0$ and $xp \neq 0$ for some $p \in F$. Therefore, $xp(p^{-1}E) = (0)$ and $p^{-1}E$ is not M-dense and $E' \subset F'$ by Lemma 9. So a decreasing sequence of annihilators in S would force an increasing sequence of rationally closed submodules of M_R, a contradiction. Therefore, ${}_SM$ satisfies the minimum condition on annihilators and $N(S)$ is nilpotent by Proposition 7. By Theorem 3 every nil subring of S is nilpotent.

COROLLARY 11. *(Lance Small). In the ring of endomorphisms of a Noetherian module a nil subring is nilpotent.*

PROOF. A noetherian module has the maximum condition on rationally closed submodules.

A ring R is said to be *semiprimary* if R is semiperfect and the Jacobson radical of R is nilpotent.

COROLLARY 12. *Let M_R be an injective module which has the maximum condition on rationally closed submodules. Then the ring of endomorphisms of M_R is a semiprimary ring.*

PROOF. This follows from Proposition 1 and Theorem 10.

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